Caringbah High School

Year 12 2023
Mathematics Extension 2
HSC Course
Assessment Task 4 – Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 100

Section I **10 marks**
Attempt Questions 1-10
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **90 marks**
Attempt Questions 11-16
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ______________________________ **Class:** ____________

Marker's Use Only							
Section I	**Section II**						**Total**
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**	**Q15**	**Q16**	
/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1. What is the quadratic equation with solutions $2+3i$ and $2-3i$?

(A) $z^2 - 4z + 13 = 0$

(B) $z^2 - 4z - 13 = 0$

(C) $z^2 + 4z - 5 = 0$

(D) $z^2 + 4z - 13 = 0$

2. Which of the following is an expression for $\int \frac{x}{\sqrt[3]{x^2+1}}\,dx$?

(A) $\frac{3}{2}\sqrt[3]{(x^2+1)^2} + C$

(B) $\frac{1}{2}\sqrt[3]{(x^2+1)^2} + C$

(C) $\frac{3}{4}\sqrt[3]{(x^2+1)^2} + C$

(D) $\frac{3}{2}\sqrt[3]{(x^2+1)^2} + C$

3. What is the size of the acute angle θ between the vectors
$\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{k}$?

(A) $\theta = \frac{\pi}{6}$

(B) $\theta = \frac{\pi}{5}$

(C) $\theta = \frac{\pi}{3}$

(D) $\theta = \frac{\pi}{4}$

4. Which of the following statements is TRUE?

(A) $\forall a, b \in \mathbb{R} \qquad \sin a < \sin b \Rightarrow a < b$

(B) $\forall a, b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \sin a < \sin b \Rightarrow a < b$

(C) $\forall a, b \in \mathbb{R} \qquad \cos a < \cos b \Rightarrow a < b$

(D) $\forall a, b \in [0, \pi] \qquad \cos a < \cos b \Rightarrow a < b$

5. Each pair of lines given below intersects at $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Which pair of lines are perpendicular?

(A) $\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and $\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix}3\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}$

(B) $\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix}2\\4\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix}-2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\3\end{pmatrix}$

(C) $\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix}0\\-3\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\-3\end{pmatrix}$ and $\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix}-1\\-2\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$

(D) $\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix}3\\-2\\8\end{pmatrix} + \lambda\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and $\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix}3\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\1\end{pmatrix}$

6. A particle moves in simple harmonic motion such that $v^2 + 9x^2 = k$.
What is the period of the particle's motion ?

(A) $\dfrac{2\pi}{k}$

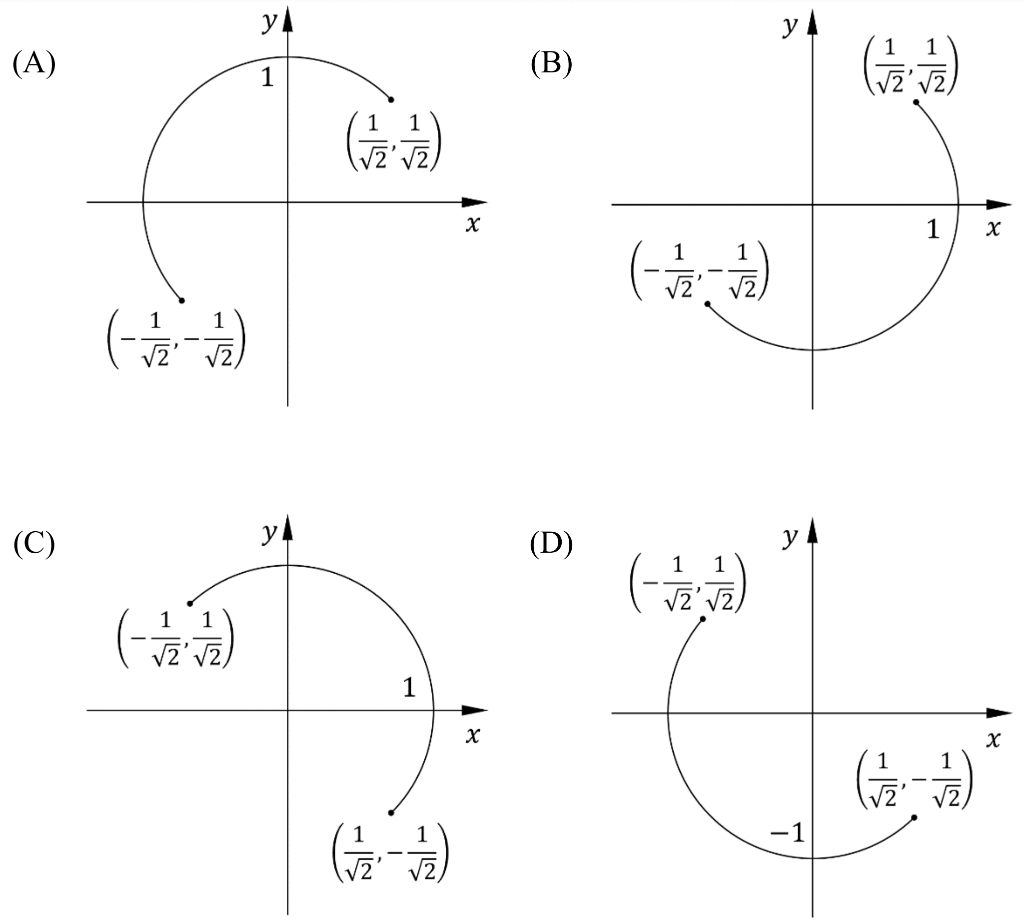
(B) 3π

(C) $\dfrac{3k}{2\pi}$

(D) $\dfrac{2\pi}{3}$

7. Which diagram best shows the curve described by the position vector

$$\underset{\sim}{r}(t) = \sin(t)\,\underset{\sim}{i} - \cos(t)\,\underset{\sim}{j} \;\text{ for }\; \frac{\pi}{4} \le t \le \frac{5\pi}{4}\,?$$



8. Which expression is equal to

$$\int \left(x^2 \frac{d}{dx}(x^3 - 1) + (x^3 - 1)\frac{d}{dx}(x^2) \right) dx\,?$$

(A) $\dfrac{x(x^2-1)}{6} + C$

(B) $\dfrac{x^3}{3} \times \dfrac{x^4 - x}{4} + C$

(C) $x^2(x^3 - 1) + C$

(D) $2x(3x^2 - 1) + C$

9. What are the values of real numbers p and q such that $(2 - i)$ is a root of the equation $z^3 + pz + q = 0$?

(A) $p = -11$ and $q = -20$

(B) $p = -11$ and $q = 20$

(C) $p = 11$ and $q = -20$

(D) $p = 11$ and $q = 20$

10. The non-zero complex numbers ω and z are linked by the formula $\omega = z + k\bar{z}$, where k is real and $k \neq 0$.

Which of the following statements is INCORRECT?

(A) ω can only be purely imaginary if $k = -1$

(B) ω can only be purely real if $k = 1$

(C) ω and z must have different moduli

(D) ω and z must have different arguments

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $z = 4 - i$, express the following in the form $a + ib$ where a and b are real.

(i) $\overline{iz}$ **1**

(ii) $\frac{1}{z}$ **1**

(b) Find the value of $\int_0^{\sqrt{5}} \frac{x}{\sqrt{x^2+4}}\,dx$. **2**

(c) The point A has position vector $\overrightarrow{OA} = 2\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}$ relative to an origin O. **2**
Find a unit vector parallel to $\overrightarrow{OA}$.

(d) **(i)** Find the values of a, b and c given **2**

$$\frac{1}{x(x^2+1)} \equiv \frac{a}{x} + \frac{bx+c}{x^2+1}.$$

(ii) Hence, or otherwise, evaluate **2**

$$\int_1^2 \frac{dx}{x(x^2+1)}$$

(e) **(i)** Find the two square roots of $2i$, giving the answers in the form $x + iy$, where x and y are real numbers. **2**

(ii) Hence, or otherwise, solve $2z^2 + 2\sqrt{2}z + 1 - i = 0$, leaving answers in the form $x + iy$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the equation $z^5 + 1 = 0$. Find the roots of this equation and show them on an Argand diagram. **3**

(b) For all non-negative numbers, x and y, $\frac{x+y}{2} \geq \sqrt{xy}$. (Do NOT prove this.) **4**

A rectangle has dimensions a and b.

Given that the rectangle has perimeter P, and area A, prove that $P^2 \geq 16A$.

(c) Use integration by parts to evaluate **3**

$$\int_1^2 xe^{2x}\, dx.$$

(d) A particle is moving in simple harmonic motion with its acceleration given by:

$$\ddot{x} = -12\sin 2t$$

Initially, the particle is at the origin and has a positive velocity of 6 m/s.

(i) Show the equation for the particle's velocity is $\dot{x} = -12\sin^2 t + 6$. **2**

(ii) Show that $\ddot{x} = -4x$. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) By completing the square, find $\int \frac{1}{9x^2 + 6x + 5}\,dx$ **3**

(b) **(i)** Show that for any integer n, $e^{in\theta} - e^{-in\theta} = 2i\sin(n\theta)$. **2**

(ii) By expanding $\left(e^{i\theta} - e^{-i\theta}\right)^3$, show that **3**

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

(c) Lines l_1 and l_2 are given below, relative to a fixed origin O.

$$l_1:\ \underset{\sim}{r} = \left(11\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k}\right) + \lambda\left(-2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}\right)$$

$$l_2:\ \underset{\sim}{r} = \left(-5\underset{\sim}{i} + 11\underset{\sim}{j} + p\,\underset{\sim}{k}\right) + \mu\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

(i) Given that line l_1 and l_2 intersect, find the value of p. **3**

(ii) Hence find the point of intersection of line l_1 and l_2. **1**

(d) Plot the following points on the Argand Plane and show that $z = 2i$, $w = \sqrt{3} - i$ and $v = -\sqrt{3} - i$ are vertices of an equilateral triangle. **3**

End of Question 13

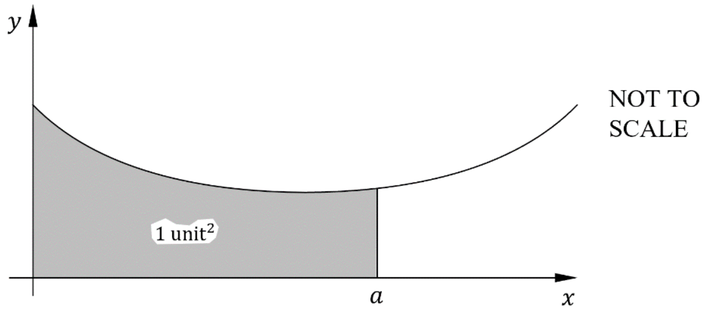
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Given that the three cube roots of i are $z_1 = e^{\frac{\pi}{6}i}, z_2 = e^{\frac{5\pi}{6}i}$ and $z_3 = -i$, prove that: **4**

$$\left(z_3\right)^2 = \left(z_1 i\right)^2 + \left(z_2 i\right)^2$$

(b) Prove that $\log_2 n$ is irrational when n is an odd integer greater than or equal to 3. **3**

(c) The area under the curve $y = \frac{1}{1+\sin x}$ from 0 to $x = a$ is 1 unit^2. **4**



Find the value of a.

(d) Show that $\frac{x}{e} > \ln x$ for $x > e$. **2**

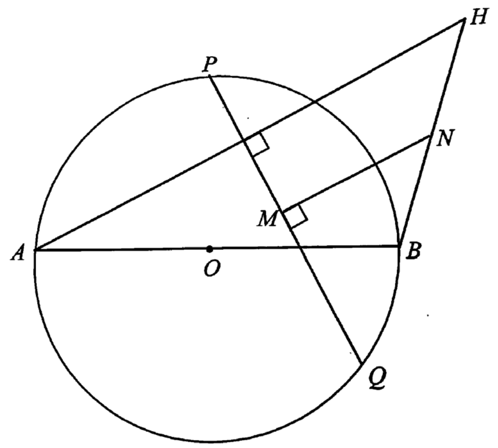
(e) A particle is moving in simple harmonic motion along the x-axis with an amplitude of 3 metres. At time t seconds it has displacement x metres from the origin O and velocity v ms^{-1} given by $v^2 = n^2(a^2 - (x-c)^2)$. **2**

If the particle has a speed $2\sqrt{5}$ ms^{-1} at the origin and a speed 6 ms^{-1} when it is 2 metres to the right of the origin, what is its centre of motion?

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a)



In the diagram, AB is the diameter of a circle with centre O and PQ is a chord of the circle that is not perpendicular to AB. The perpendicular from A to PQ is produced to the point H outside the circle. M is the midpoint of PQ and N is the point on BH such that $MN \perp PQ$ and the points O, M and N are collinear.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OP} = \underset{\sim}{p}$, $\overrightarrow{OQ} = \underset{\sim}{q}$ and $\overrightarrow{AH} = \underset{\sim}{h}$.

(i) If $\overrightarrow{BN} = \lambda\ \overrightarrow{BH}$ for some scalar λ, show $\overrightarrow{ON}$ is $(2\lambda - 1)\underset{\sim}{a} + \lambda \underset{\sim}{h}$. **1**

(ii) Hence show that N is the midpoint of BH. **3**

(b) Let $z = x + iy$ and hence solve the equation $z^2 = |z|^2 - 4$ **3**

(c) Position vectors of the points *A*, *B* and *C*, relative to an origin *O*, are $-\underset{\sim}{i} - \underset{\sim}{j}$, $\underset{\sim}{j} + 2\underset{\sim}{k}$, and $4\underset{\sim}{i} + \underset{\sim}{k}$ respectively.

(i) Find $\overrightarrow{AB}$. **1**

(ii) Find $|\overrightarrow{AB}|$. **1**

(iii) Prove that $\angle ABC$ is a right angle. **3**

(d) A particle of mass *m* is moving in a straight line under the action of a force. **3**

$$F = \frac{m}{x^3}(6 - 10x)$$

What is the velocity in any position if the particle starts from rest at $x = 1$?

End of Question 15

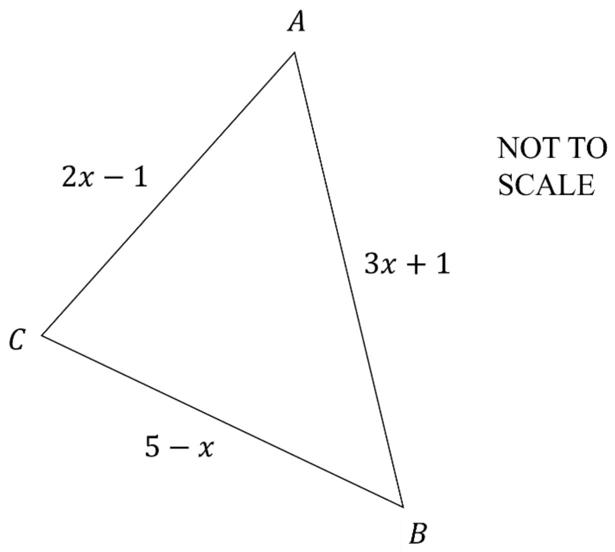
Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) **(i)** Let $I_n = \int_0^1 (1 - x^r)^n dx$ where $r > 0$ for $n = 1, 2, 3, \ldots$

Show that $I_n = \dfrac{nr}{nr + 1} I_{n-1}$ **2**

(ii) Hence or otherwise, find the exact value of $\int_0^1 (1 - x^{\frac{3}{2}})^3 dx$ **2**

(b) Triangle ABC has sides $a = 5 - x,\ b = 2x - 1$ and $c = 3x + 1$ as shown, where $x \in \mathbb{R}$.



(i) Prove that $\frac{5}{6} < x < \frac{3}{2}$. **2**

(ii) Given ΔABC is isosceles, prove that $\cos A = \frac{1}{8}$. **2**

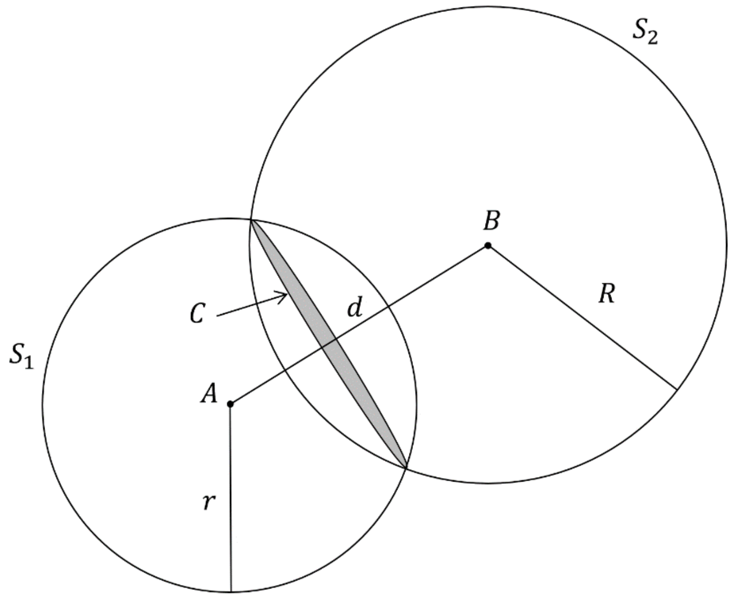
(c) Sketch the locus of z on the Argand diagram of the below and mark where the inequalities $|z - 1| \leq 3$ and $\text{Im}(z) \geq 3$ hold simultaneously. **2**

Question 16 continues next page

(d) Two spheres, S_1 and S_2, intersect in the circle C.

The sphere S_1 is centred at $A(a, b, c)$ with radius r and the sphere S_2 is centred at $B(a + m, b + n, c + p)$ with radius R, where $a,\ b,\ c,\ m,\ n,\ p,\ r$ and R are real, with $r,\ R > 0$.

The distance between the centres of the spheres is d.



(i) Prove that the circle C lies on the sphere S_3, which has AB as a diameter, only if $m^2 + n^2 + p^2 = r^2 + R^2$. **4**

(ii) Hence, or otherwise, show that the intersection of the spheres $x^2 + y^2 + z^2 = 25$ and $(x - 3)^2 + (y - 4)^2 + (z - 12)^2 = 144$ is a circle lying on a third sphere, which has the interval joining the centres of the first two spheres as a diameter. **1**

End of Examination

Caringbah High School

Year 12 2023

Mathematics Extension 2

HSC Course

Assessment Task 4 – Trial HSC Examination Solutions

Section I

Multiple-choice Answer Key

Question	Answer
1	A
2	C
3	A
4	B
5	A
6	D
7	C
8	C
9	B
10	D

Worked Solutions

1 A

$(z-(2+3i))(z-(2-3i))=0$

$z^2-2z+3iz-2z-3iz+4+9=0$

$\therefore z^2-4z+13=0$, so A.

2 C

Let $u=x^2+1$

$\dfrac{du}{dx}=2x$ and $xdx=\dfrac{1}{2}du$

$$\begin{aligned}\int\frac{x}{\sqrt[3]{x^2+1}}dx &= \frac{1}{2}\int\frac{1}{\sqrt[3]{u}}du\\ &= \frac{1}{2}\int u^{-\frac{1}{3}}du\\ &= \frac{1}{2}\times\frac{3}{2}u^{\frac{2}{3}}+C\\ &= \frac{3}{4}\sqrt[3]{(x^2+1)^2}+C\end{aligned}$$

, so C.

3 A

$\cos\theta=\dfrac{\underset{\sim}{a}\,.\,\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}=\dfrac{4+0+2}{\sqrt{6}\sqrt{8}}=\dfrac{3}{2\sqrt{3}}=\dfrac{\sqrt{3}}{2}\qquad\therefore\theta=\dfrac{\pi}{6}$, so A.

4 B

The correct answer must be an increasing function in the whole domain, which only occurs for B.

5 A

Looking for the dot product of the direction vectors being zero.

A: $\begin{pmatrix}-1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\0\end{pmatrix}=-2+2+0=0$, so A.

6 D

$v^2 + 9x^2 = k$

$v^2 = k - 9x^2$

$$\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{k}{2} - \frac{9x^2}{2}\right)$$

$$= -9x = -n^2 x$$

Hence $n = 3$

Period: $T = \frac{2\pi}{n} = \frac{2\pi}{3}$, so D.

7 C

$$\underset{\sim}{r}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} \times \underset{\sim}{i} - \cos\frac{\pi}{4} \times \underset{\sim}{j} = \frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{1}{\sqrt{2}}\underset{\sim}{j} = \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$$

$$\underset{\sim}{r}\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} \times \underset{\sim}{i} - \cos\frac{5\pi}{4} \times \underset{\sim}{j} = -\frac{1}{\sqrt{2}}\underset{\sim}{i} - \left(-\frac{1}{\sqrt{2}}\right)\underset{\sim}{j} = \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

Also passes through:

$$\underset{\sim}{r}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} \times \underset{\sim}{i} - \cos\frac{\pi}{2} \times \underset{\sim}{j} = \underset{\sim}{i} = (1,\ 0),$$ so C.

8 C

The integrand involves the product rule

$$\int \left(x^2 \frac{d}{dx}(x^3 - 1) + (x^3 - 1)\frac{d}{dx}(x^2)\right) dx$$

$$= \int \frac{d}{dx}\left(x^2(x^3 - 1)\right) dx$$

$= x^2(x^3 - 1) + c,$ so C.

Alternatively:

$$\int \left(x^2 \frac{d}{dx}(x^3 - 1) + (x^3 - 1)\frac{d}{dx}(x^2)\right) dx$$

$$= \int \left(x^2 \times 3x^2 + (x^3 - 1) \times 2x\right) dx$$

$$= \int \left(3x^4 + 2x^4 - 2x\right) dx = \int (5x^4 - 2x) dx$$

$$= x^5 - x^2 + c$$

$= x^2(x^3 - 1) + c,$ so C.

9 B

Using the conjugate root theorem $(2+i)$ and $(2-i)$ are both roots of the equation $z^3+pz+q=0$.

$(2+i)+(2-i)+\alpha=0$ (sum of the roots)

$\alpha=-4$

$(2+i)\times(2-i)\times(-4)=-q$ (product of the roots)

$(4+1)\times(-4)=-q$

$q=20$

$(2+i)(2-i)+(2-i)\times(-4)+(2+i)\times(-4)=p$

$\therefore p=-11$ and $q=20$, so B.

10 D

A: when $k=-1$, $z-1(\bar{z})=2\text{Im}(z)$ which is purely imaginary

B: when $k=1$, $z+1(\bar{z})=2\text{Re}(z)$ which is purely real.

C: the modulus of z is $\sqrt{x^2+y^2}$ while the modulus of ω is $\sqrt{(k+1)^2x^2+(k-1)^2y^2}$, and the two moduli could only be equal if $k+1=1\ \rightarrow k=0$ <u>and</u> $k-1=1\rightarrow k=2$, which is a contradiction.

D: when z is purely real then its argument is 0 or π, and its conjugate, $\bar{z}$, is equal so has the same argument. When added together the argument is unchanged, so ω has the same argument and so D is INCORRECT.

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\overline{iz} = \overline{i(4-i)} = \overline{4i+1}$$
$$= 1 - 4i \quad \boxed{A}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\frac{1}{z} = \frac{1}{4-i} \times \frac{4+i}{4+i} = \frac{4+i}{16+1}$$

$$= \frac{4}{17} + \frac{1}{17}i \quad or \quad \frac{4+i}{17} \quad \boxed{A}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Sets up the integral in terms of u.	1

Sample answer:

Let $u = x^2 + 4$

$\frac{du}{dx} = 2x \; or \; \frac{1}{2}du = xdx$

When $x = 0$ then $u = 4$ and when $x = \sqrt{5}$ then $u = 9$

$$\int_0^{\sqrt{5}} \frac{x}{\sqrt{x^2+4}}dx = \frac{1}{2}\int_4^9 \frac{1}{\sqrt{u}}du = \frac{1}{2}\int_4^9 u^{-\frac{1}{2}}du \quad \boxed{A}$$

$$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_4^9$$

$$= \left[\sqrt{9} - \sqrt{4}\right]$$

$$= 1 \quad \boxed{B}$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	2
• Finds the magnitude of $\overrightarrow{OA}$.	1

Sample answer:

$$\left|\overrightarrow{OA}\right| = \sqrt{2^2 + 6^2 + (-3)^2}$$

$$= \sqrt{49} = 7 \quad \boxed{A}$$

$$\widehat{OA} = \frac{\overrightarrow{OA}}{\left|\overrightarrow{OA}\right|}$$

$$= \frac{1}{7}(2\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}) \quad \boxed{B}$$

Question 11 (d) (i)

Criteria	Marks
• Provides correct solution	2
• Rearranges the numerator as $(a+b)x^2+cx+a$, or equivalent merit	1

Sample answer:

$$\frac{1}{x(x^2+1)} \equiv \frac{a}{x} + \frac{bx+c}{x^2+1}$$

$$\equiv \frac{a(x^2+1)+x(bx+c)}{x(x^2+1)}$$

$$\equiv \frac{(a+b)x^2+cx+a}{x(x^2+1)} \quad \boxed{A}$$

Equating coefficients $a+b=0,\ c=0,\ a=1$ and thus $b=-1$

$a=1, b=-1, c=0$ $\boxed{B}$

Question 11 (d) (ii)

Criteria	Marks
• Provides correct solution	2
• Find the correct primitive, or equivalent merit	1

Sample answer:

$$\int_1^2 \frac{dx}{x(x^2+1)} = \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$$

$$= \int_1^2 \left(\frac{1}{x} - \frac{1}{2} \times \frac{2x}{x^2+1}\right) dx$$

$$= \left[\ln|x| - \frac{1}{2}\ln|x^2+1|\right]_1^2 \quad \boxed{A}$$

$$= \left(\ln 2 - \frac{1}{2}\ln 5\right) - \left(\ln 1 - \frac{1}{2}\ln 2\right)$$

$$= \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5 \ \left(\text{or } \ln\left(\frac{2\sqrt{2}}{\sqrt{5}}\right)\right) \quad \boxed{B}$$

Question 11 (e) (i)

Criteria	Marks
• Provides correct solution	2
• Finds one square root, or finds square roots in another form, or equivalent merit	1

Sample answer:

$$(a+ib)^2 = 2i$$

$$a^2 - b^2 + 2abi = 2i$$

$$a^2 - b^2 = 0 \quad \text{and} \quad ab = 1$$

$a = \pm 1,\ b = \pm 1$ by inspection $\boxed{A}$

$\therefore$ the two roots are $\pm(1+i)$ $\boxed{B}$.

Question 11 (e) (ii)

Criteria	Marks
• Provides correct solution	3
• Simplifies the root of the discriminant to $\sqrt{8i}$ or equivalent merit	2
• Correctly applies the quadratic formula, or equivalent merit	1

Sample answer:

$$z = \frac{-2\sqrt{2} \pm \sqrt{\left(2\sqrt{2}\right)^2 - 4(2)(1-i)}}{2(2)} \qquad \boxed{A}$$

$$= \frac{-2\sqrt{2} \pm \sqrt{8 - 8 + 8i}}{4}$$

$$= \frac{-2\sqrt{2} \pm 2\sqrt{2i}}{4} \qquad \boxed{B}$$

$$= \frac{-\sqrt{2} \pm \sqrt{2i}}{2}$$

$$= \frac{-\sqrt{2} \pm (1+i)}{2}$$

$$= -\frac{\sqrt{2}+1}{2} - \frac{i}{2},\ \frac{1-\sqrt{2}}{2} + \frac{i}{2} \qquad \boxed{C}$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	3
• Correctly finds all the roots, or equivalent merit	2
• Correctly finds one root, or equivalent merit	1

Sample answer:

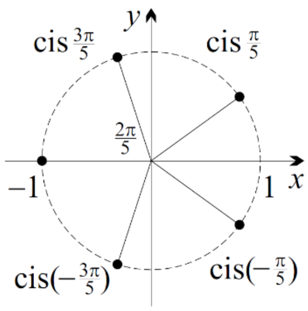
$z^5 + 1 = 0 \quad \therefore z^5 = -1$

$Let\ z = cis\theta \quad \therefore z^5 = cis5\theta = -1$

$\therefore 5\theta = \pi \qquad \therefore \theta = \frac{\pi}{5}$

$\therefore z_1 = cis\frac{\pi}{5}$ $\boxed{A}$

$\therefore z_2 = cis\frac{3\pi}{5}, z_3 = -1, z_4 = cis\frac{-\pi}{5}, z_5 = cis\frac{-3\pi}{5}$ $\boxed{B}$



$\boxed{C}$

Question 12 (b)

Criteria	Marks
• Provides correct solution	4
• Correctly replaces A and P into the AM-GM	3
• Substitutes a and b into the AM-GM, or equivalent merit	2
• Finds expressions for the perimeter and area in terms of a and b, or equivalent merit	1

Sample answer:

$P = 2(a+b), A = ab$ $\boxed{A}$

$$\frac{a+b}{2} \geq \sqrt{ab} \qquad \boxed{B}$$

$$\frac{2(a+b)}{4} \geq \sqrt{ab}$$

$$\frac{P}{4} \geq \sqrt{A} \qquad \boxed{C}$$

$$\frac{P^2}{16} \geq A$$

$$P^2 \geq 16A \qquad \boxed{D}$$

Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Finds the integral equals $\frac{2e^4}{2} - \frac{e^2}{2} - \frac{1}{2}\left[\frac{1}{2}e^{2x}\right]_1^2$	2
• Differentiates one function and integrates the other	1

Sample answer:

$$\int_1^2 xe^{2x}\,dx \qquad \boxed{\begin{array}{ll} u = x & \frac{dv}{dx} = e^{2x} \\ \frac{du}{dx} = 1 & v = \frac{1}{2}e^{2x} \end{array}} \qquad \boxed{A}$$

$$= \left[\frac{xe^{2x}}{2}\right]_1^2 - \frac{1}{2}\int_1^2 e^{2x}\,dx$$

$$= \frac{2e^4}{2} - \frac{e^2}{2} - \frac{1}{2}\left[\frac{1}{2}e^{2x}\right]_1^2 \qquad \boxed{B}$$

$$= e^4 - \frac{e^2}{2} - \frac{e^4 - e^2}{4}$$

$$= \frac{3e^4}{4} - \frac{e^2}{4} \qquad \boxed{C}$$

$$= \frac{e^2(3e^2 - 1)}{4}$$

Question 12 (d) (i)

Criteria	Marks
• Provides correct solution	2
• Finds velocity but does not clear constant, or equivalent merit	1

Sample answer:

$$\ddot{x} = -12\sin 2t$$
$$\dot{x} = -12\int \sin 2t\,dt$$
$$= -12\int 2\sin t\cos t\,dt$$
$$= -12\sin^2 t + C \quad \boxed{A}$$
When $t = 0,\ \dot{x} = 6$ hence $C = 6$
$$\therefore \dot{x} = -12\sin^2 t + 6 \quad \boxed{B}$$

OR

$$\dot{x} = -12\int \sin 2t\,dt$$
$$= -12\left[-\frac{\cos 2t}{2}\right] + \mathrm{C}$$
$$= 6\cos 2\mathrm{t} + C \quad \boxed{A}$$
When $t = 0,\ \dot{x} = 6$ hence $C = 0$
$$\therefore \dot{x} = 6\cos 2t = 6(1 - 2\sin^2 t)$$
$$= -12\sin^2 t + 6 \quad \boxed{B}$$

Question 12 (d) (ii)

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards the solution.	2
• Finds $x = 3\sin 2t + C$ or equivalent merit.	1

Sample answer:

$$x = \int -12\sin^2 t + 6dt$$
$$= \int -12\left(\frac{1-\cos 2t}{2}\right) + 6dt = \int -6 + 6\cos 2t + 6dt$$
$$= \int 6\cos 2t dt$$
$$= 3\sin 2t + C \quad \boxed{A}$$
When $t = 0, x = 0$ hence $C = 0$
$$\therefore x = 3\sin 2t \quad \boxed{B}$$
Then
$$\ddot{x} = -12\sin 2t$$
$$= -4 \times 3\sin 2t$$
$$= -4x \quad \boxed{C}$$

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards the solution.	2
• Completes the square or equivalent merit.	1

Sample answer:

$$\int \frac{1}{9x^2+6x+5}\,dx = \int \frac{dx}{9\left[\left(x^2+\frac{2}{3}x\right)\right]+5}$$

$$= \int \frac{dx}{9\left[\left(x+\frac{1}{3}\right)^2-\frac{1}{9}\right]+5} \quad \boxed{A}$$

$$= \int \frac{dx}{9\left(x+\frac{1}{3}\right)^2+4}$$

$$= \int \frac{dx}{(3x+1)^2+2^2} \quad \boxed{B}$$

$$= \frac{1}{3}\int \frac{3dx}{(3x+1)^2+2^2}$$

$$= \frac{1}{6}\tan^{-1}\frac{(3x+1)}{2}+C \quad \boxed{C}$$

Question 13 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Rewrites as the difference of conjugates or converts into mod-arg form	1

Sample answer:

$$e^{in\theta} - e^{-in\theta} = e^{in\theta} - \overline{e^{in\theta}} \quad \boxed{A}$$
$$= 2\mathrm{Im}\left(e^{in\theta}\right)$$
$$= 2i\sin n\theta \quad \boxed{B}$$

Alternative solution:

$$e^{in\theta} - e^{-in\theta} = (\cos n\theta + i\sin n\theta) - (\cos(-n\theta) + i\sin(-n\theta)) \quad \boxed{A}$$
$$= (\cos n\theta + i\sin n\theta) - (\cos(n\theta) - i\sin(n\theta))$$
$$= 2i\sin n\theta \quad \boxed{B}$$

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds both equations marked as (1) and (2), or equivalent merit	2
• Correctly uses the binomial expansion (without simplifying) or uses the result in (i) to show $\left(e^{i\theta} - e^{-i\theta}\right)^3 = (2i \sin\theta)^3$, or equivalent merit	1

Sample answer:

$$\left(e^{i\theta} - e^{-i\theta}\right)^3 = \left(e^{i\theta}\right)^3 + 3\left(e^{i\theta}\right)^2\left(-e^{-i\theta}\right) + 3\left(e^{i\theta}\right)\left(-e^{-i\theta}\right)^2 + \left(-e^{-i\theta}\right)^3 \quad \boxed{A}$$

$$= e^{i3\theta} - 3e^{-i\theta} + 3\,e^{-i\theta} - e^{-i3\theta}$$

$$= \left(e^{i3\theta} - e^{-i3\theta}\right) - 3\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= 2i\sin 3\theta - 6i\sin\theta \qquad (1)$$

From (i):

$$\left(e^{i\theta} - e^{-i\theta}\right)^3 = (2i\sin\theta)^3$$

$$= 8i^3\sin^3\theta$$

$$= -8\sin^3\theta \qquad (2) \qquad \boxed{B} \text{ or } \boxed{A} \left(\text{without (1)}\right)$$

From (1) and (2):

$$-8\sin^3\theta = 2\sin 3\theta - 6\sin\theta$$

$$\sin^3\theta = \frac{3\sin\theta + \sin 3\theta}{4} \qquad \boxed{C}$$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Finds the correct values for λ and μ, or equivalent merit	2
• Correctly uses the binomial expansion (without simplifying) or uses the result in (i) to show $\left(e^{i\theta} - e^{-i\theta}\right)^3 = (2\sin\theta)^3$, or equivalent merit	1

Sample answer:

Line l_1 intersects the line l_2 then:

$$\left(11\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k}\right) + \lambda\left(-2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}\right)$$
$$= \left(-5\underset{\sim}{i} + 11\underset{\sim}{j} + p\underset{\sim}{k}\right) + \mu\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right) \quad \boxed{A}$$

$$11 - 2\lambda = -5 - 3\mu \quad ①$$
$$2 + \lambda = 11 + 2\mu \quad ②$$
$$17 - 4\lambda = p + 2\mu \quad ③$$

Equation ① + 2 × ② $15 = 17 + \mu$ then $\mu = -2$

Equation ② $2 + \lambda = 11 - 4$ then $\lambda = 5$ $\boxed{B}$

Equation ③ $17 - 4 \times 5 = p + 2 \times -2$

$$p = 17 - 20 + 4$$
$$= 1$$

∴ The value of p is 1. $\boxed{C}$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

From 13(b)(i) $\lambda = 5$, $\mu = -2$ and $p = 1$

l_1: $\left(11\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k}\right) + 5\left(-2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}\right) = \underset{\sim}{i} + 7\underset{\sim}{j} + -3\underset{\sim}{k}$ or

l_2: $\left(-5\underset{\sim}{i} + 11\underset{\sim}{j} + \underset{\sim}{k}\right) + -2\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right) = \underset{\sim}{i} + 7\underset{\sim}{j} + -3\underset{\sim}{k}$

∴ Point of intersection is $\underset{\sim}{i} + 7\underset{\sim}{j} + -3\underset{\sim}{k}$ $\boxed{A}$

Question 13 (d)

Criteria	Marks
• Provides correct solution	3
• Finds the length of zw or zv or wv, or equivalent merit	2
• Plots the points or equivalent merit	1

Sample answer:

Pythagoras theorem

$zw = \sqrt{3^2 + (\sqrt{3})^2}$ $\boxed{B}$

$= \sqrt{12} = 2\sqrt{3}$

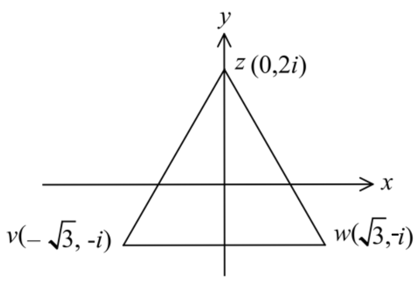
$zv = \sqrt{3^2 + (\sqrt{3})^2}$

$= \sqrt{12} = 2\sqrt{3}$

$wv = \sqrt{3} + \sqrt{3}$

$= 2\sqrt{3}$

$\therefore$ Equilateral triangle. $\boxed{C}$



$\boxed{A}$

Question 14 (a)

Criteria	Marks
• Provides correct solution	4
• Proves $\left(z_1 i\right)^2 + \left(z_2 i\right)^2 = -1$, or $\left(z_1\right)^2 + \left(z_2\right)^2 + \left(z_3\right)^2 = 0$, or equivalent merit	3
• Uses conjugate rules or mod-arg form to simplify the sum of two complex numbers, or equivalent merit	2
• Uses de Moivres Theorem, or equivalent merit	1

Sample answer:

$$\left(z_1 i\right)^2 + \left(z_2 i\right)^2 = i^2\left(\left(e^{\frac{\pi}{6}i}\right)^2 + \left(e^{\frac{5\pi}{6}i}\right)^2\right)$$

$$= -\left(e^{\frac{\pi}{3}i} + e^{\frac{5\pi}{3}i}\right) \qquad \boxed{A}$$

$$= -e^{i\pi}\left(e^{-\frac{2\pi}{3}i} + e^{\frac{2\pi}{3}i}\right)$$

$$= -(-1)\left(e^{\frac{2\pi}{3}i} + \overline{e^{\frac{2\pi}{3}i}}\right)$$

$$= 1 \times 2\mathrm{Re}\left(e^{\frac{2\pi}{3}i}\right) \qquad \boxed{B}$$

$$= 2\cos\left(\frac{2\pi}{3}\right)$$

$$= 2\left(-\frac{1}{2}\right)$$

$$= -1 \qquad \boxed{C}$$

$$= (-i)^2$$

$$= \left(z_3\right)^2 \qquad \boxed{D}$$

Question 14 (b)

Criteria	Marks
• Provides correct solution	3
• Rearranges to find $(2k+1)^q = 2^p$, or equivalent merit	2
• Defines $\log_2(2k+1) = \frac{p}{q}$, or equivalent merit	1

Sample answer:

Suppose $\log_2 n$ is rational.

Let $n = 2k+1$ for $k \in \mathbb{Z}^+$

$\therefore \log_2(2k+1) = \frac{p}{q}$ for $p,\ q \in \mathbb{Z},\ HCF(p,q) = 1,\ q \neq 0$ $\boxed{A}$

$\therefore q\log_2(2k+1) = p$

$\log_2(2k+1)^q = p$

$(2k+1)^q = 2^p$ $\boxed{B}$

LHS is odd, since an odd number to any power is odd *.

RHS is even, since an even number to any power is even *.

We have a contradiction, so $\log_2 n$ must be irrational for all odd n. $\boxed{C}$

* Must mention that an odd number to any power is odd, not just LHS is odd, and similarly for the RHS being even.

Question 14 (c)

Criteria	Marks
• Provides correct solution	4
• Finds the value of the integral as $2\left(1-\frac{1}{1+\tan\frac{a}{2}}\right)$ or equivalent	3
• Simplifies the integrand to $(1+t)^{-2}$	2
• Correctly substitutes for t including changing the limits and replacing dx	1

Sample answer:

$$\int_0^a \frac{dx}{1+\sin x} \qquad t=\tan\frac{x}{2} \ \rightarrow \ dx=\frac{2dt}{1+t^2}$$

$$=\int_0^{\tan\frac{a}{2}} \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2} \qquad \boxed{A}$$

$$=2\int_0^{\tan\frac{a}{2}} \frac{dt}{1+t^2+2t}$$

$$=2\int_0^{\tan\frac{a}{2}} (1+t)^{-2}\,dt \qquad \boxed{B}$$

$$=2\left[\frac{(1+t)^{-1}}{-1}\right]_0^{\tan\frac{a}{2}} = 2\left[\frac{1}{1+t}\right]_{\tan\frac{a}{2}}^{0}$$

$$=2\left(1-\frac{1}{1+\tan\frac{a}{2}}\right) \qquad \boxed{C}$$

$$\therefore 2\left(1-\frac{1}{1+\tan\frac{a}{2}}\right)=1$$

$$1-\frac{1}{1+\tan\frac{a}{2}}=\frac{1}{2}$$

$$\frac{1}{1+\tan\frac{a}{2}}=\frac{1}{2}$$

$$1+\tan\frac{a}{2}=2$$

$$\tan\frac{a}{2}=1$$

$$\frac{a}{2}=\frac{\pi}{4} \qquad \therefore a=\frac{\pi}{2} \qquad \boxed{D}$$

Question 14 (d)

Criteria	Marks
• Provides correct solution	2
• Uses $f(e) = 0$ to deduce required inequality.	1

Sample answer:

Let $f(x) = \frac{x}{e} - \ln x$ for $x > e$

Hence $f(e) = 0$ and $f(x)$ is an increasing function $x > e$ $\boxed{A}$

$\therefore f(x) > 0$ for $x > e$

$\therefore \frac{x}{e} > \ln x$ for $x > e$ $\boxed{B}$

Question 14 (e)

Criteria	Marks
• Provides correct solution	2
• Write a simultaneous equation, or equivalent merit	1

Sample answer:

$x = 0, \quad v^2 = 20 \quad \therefore n^2(9 - c^2) = 20 \quad -①$

$x = 2, \quad v^2 = 36 \quad \therefore n^2(9 - (2 - c)^2) = 36 \quad -②$ $\boxed{A}$

$\therefore \frac{①}{②} = \frac{(9 - c^2)}{(9 - (2 - c)^2)} = \frac{20}{36}$

$\therefore \quad \frac{(9 - c^2)}{(5 + 4c - c^2)} = \frac{5}{9}$

$\therefore c^2 + 5c - 14 = 0$

$\therefore (c - 2)(c + 7) = 0, \quad \therefore c = 2 \; (as\; c > 0)$ $\boxed{B}$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN} = -\underset{\sim}{a} + \lambda\ \overrightarrow{BH} = -\underset{\sim}{a} + \lambda\left(\overrightarrow{BA} + \overrightarrow{AH}\right) = -\underset{\sim}{a} + \lambda\left(2\underset{\sim}{a} + \underset{\sim}{h}\right)$

$\therefore \overrightarrow{ON} = -\underset{\sim}{a} + \lambda\left(2\underset{\sim}{a} + \underset{\sim}{h}\right) = (2\lambda - 1)\underset{\sim}{a} + \lambda\underset{\sim}{h}$ $\boxed{A}$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution.	3
• Substantive progress ie. correct procedure but fails to explain $2\lambda - 1 = 0$.	2
• Some progress ie. uses the perpendicularity to write an appropriate dot product.	1

Sample answer:

$MN \perp PQ \ \therefore \underset{\sim}{h}.\left(\underset{\sim}{q} - \underset{\sim}{p}\right) = 0$ $\boxed{A}$

and O, M and N are collinear $\therefore \overrightarrow{ON}\ .\overrightarrow{PQ} = 0$ since $MN \perp PQ$

$\therefore \left((2\lambda - 1)\underset{\sim}{a} + \lambda\underset{\sim}{h}\right).\left(\underset{\sim}{q} - \underset{\sim}{p}\right) = 0$

$\therefore (2\lambda - 1)\underset{\sim}{a}.\left(\underset{\sim}{q} - \underset{\sim}{p}\right) = 0$

But AB , PQ are not perpendicular. $\boxed{B}$

$\therefore \underset{\sim}{a}.\left(\underset{\sim}{q} - \underset{\sim}{p}\right) \neq 0$

Hence $(2\lambda - 1) = 0 \quad \therefore \lambda = \frac{1}{2}$

$\therefore \overrightarrow{BN} = \frac{1}{2}\overrightarrow{BH}$ and hence N is the midpoint of BH $\boxed{C}$

Question 15 (b)

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards the solution.	2
• Writes the equation using z^2in terms of $x + iy$ or equivalent merit.	1

Sample answer:

Let $z = x + iy$

Then $z^2 = (x + iy)^2 = x^2 + 2ixy - y^2$ $\boxed{A}$

$z^2 = |z|^2 - 4$

$x^2 + 2ixy - y^2 = |x + iy|^2 - 4$

$x^2 + 2ixy - y^2 = \left(\sqrt{x^2 + y^2}\right)^2 - 4$

$x^2 + 2ixy - y^2 = x^2 + y^2 - 4$

$2ixy - y^2 = y^2 - 4$

$2y^2 - 2ixy - 4 = 0$ $\boxed{B}$

$2(y^2 - ixy - 2) = 0$

$(y^2 - 2) = 0$ then $y = \pm\sqrt{2}$

$ixy = 0$ then $x = 0$

Therefore

$z = x + iy$

$z = \pm\sqrt{2}i$ $\boxed{C}$

Question 15 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (\underset{\sim}{j} + 2\underset{\sim}{k}) - (-\underset{\sim}{i} - \underset{\sim}{j}) \\ &= \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k} \quad \boxed{A}\end{aligned}$$

Question 15 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{1^2 + 2^2 + 2^2} \\ &= 3 \quad \boxed{A}\end{aligned}$$

Question 15 (c) (iii)

Criteria	Marks
• Provides correct solution	3
• Uses the angle between two vectors.	2
• Finds $\overrightarrow{BC}$ or $\lvert\overrightarrow{BC}\rvert$.	1

Sample answer:

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB} \\ &= (4\underset{\sim}{i} + \underset{\sim}{k}) - (\underset{\sim}{j} + 2\underset{\sim}{k}) \\ &= 4\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k} \qquad \boxed{A}\end{aligned}$$

$$\begin{aligned}\lvert\overrightarrow{BC}\rvert &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{4^2 + (-1)^2 + (-1)^2} \\ &= 3\sqrt{2}\end{aligned}$$

$$\begin{aligned}\cos\angle CAB &= \frac{\overrightarrow{AB} \cdot \overrightarrow{BC}}{\lvert\overrightarrow{AB}\rvert\lvert\overrightarrow{BC}\rvert} \qquad \boxed{B} \\ &= \frac{1 \times 4 + 2 \times (-1) + 2 \times (-1)}{3 \times 3\sqrt{2}} \\ &= \frac{0}{9\sqrt{2}} = 0 \\ \angle CAB &= 90^\circ \qquad \boxed{C}\end{aligned}$$

Question 15 (d)

Criteria	Marks
• Provides correct solution	3
• Integrates and uses the initial conditions to find an expression for $\frac{1}{2}v^2$.	2
• Uses $v\frac{dv}{dx}$ for acceleration.	1

Sample answer:

$$F = \frac{m}{x^3}(6 - 10x)$$

$$ma = \frac{m}{x^3}(6 - 10x)$$

$$v\frac{dv}{dx} = \frac{6}{x^3} - \frac{10}{x^2} \qquad \boxed{A}$$

$$\int v dv = \int \left(\frac{6}{x^3} - \frac{10}{x^2}\right) dx$$

$$\frac{1}{2}v^2 = \left(\frac{6x^{-2}}{-2} - \frac{10x^{-1}}{-1}\right) + C$$

$$\frac{1}{2}v^2 = \left(\frac{-3}{x^2} + \frac{10}{x}\right) + C$$

When $v = 0$ and $x = 1$

$$\frac{1}{2}0^2 = \left(\frac{-3}{1^2} + \frac{10}{1}\right) + C$$

$$C = -7 \qquad \boxed{B}$$

Hence

$$\frac{1}{2}v^2 = \left(\frac{-3}{x^2} + \frac{10}{x}\right) - 7$$

$$v^2 = \left(\frac{-6}{x^2} + \frac{20}{x}\right) - 14$$

$$= \frac{-6 + 20x - 14x^2}{x^2}$$

$$v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2)} \qquad \boxed{C}$$

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Sets up the integration by parts.	1

Sample answer:

$$I_n = \int_0^1 (1 - x^r)^n dx$$

$$= [x(1 - x^r)^n]_0^1 - n\int_0^1 x(1 - x^r)^{n-1}(-rx^{r-1})dx \quad \boxed{A}$$

$$= 0 - nr\int_0^1 [(1 - x^r)^n - 1](1 - x^r)^{n-1}dx$$

$$= 0 - nr\int_0^1 (1 - x^r)^n - (1 - x^r)^{n-1}dx$$

$$I_n = nr(-I_n + I_{n-1})$$

$$(nr + 1)I_n = nrI_{n-1}$$

$$I_n = \frac{nr}{nr + 1}I_{n-1} \quad \boxed{B}$$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly equates I_1, or equivalent merit.	1

Sample answer:

For $r = \frac{3}{2}$ and $n = 3$

$$I_3 = \frac{3 \times \frac{3}{2}}{3 \times \frac{3}{2} + 1} \times I_2 \quad I_2 = \frac{2 \times \frac{3}{2}}{2 \times \frac{3}{2} + 1} \times I_1 \quad I_1 = \frac{1 \times \frac{3}{2}}{1 \times \frac{3}{2} + 1} \times I_0 \quad \boxed{A}$$

But $I_0 = \int_0^1 (1 - x^r)^0 dx = \int_0^1 1dx = 1$

$$I_3 = \frac{3 \times \frac{3}{2}}{3 \times \frac{3}{2} + 1} \times \frac{2 \times \frac{3}{2}}{2 \times \frac{3}{2} + 1} \times \frac{1 \times \frac{3}{2}}{1 \times \frac{3}{2} + 1} \times 1$$

$$= \frac{81}{220} \quad \boxed{B}$$

Question 16 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly uses the triangle inequality once to find either $x > \frac{5}{6}$, $x < \frac{3}{2}$ or $6 > -1$	1

Sample answer:

$2x - 1 + 3x + 1 > 5 - x$

$6x > 5$

$x > \frac{5}{6}$ (1) $\boxed{A}$ (or below)

$2x - 1 + 5 - x > 3x + 1$

$3 > 2x$

$x < \frac{3}{2}$ (2) $\boxed{A}$ (or above)

$3x + 1 + 5 - x > 2x - 1$

$6 > -1$ ie true regardless of the value of x $\boxed{A}$ (or above)

From (1) and (2):

$\frac{5}{6} < x < \frac{3}{2}$ $\boxed{B}$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Shows that $x = 1$ is the only answer that satisfies (i)	1

Sample answer:

If $2x - 1 = 3x + 1 \rightarrow x = -2$ which is outside the domain from (i) so $2x - 1 \neq 3x + 1$
If $2x - 1 = 5 - x \rightarrow x = 2$ which is outside the domain from (i) so $2x - 1 \neq 5 - x$

If $5 - x = 3x + 1 \rightarrow x = 1$ which is in the domain from (i) $\boxed{A}$

When $x = 1,\ 5 - x = 3x + 1 = 4$ and $2x - 1 = 1$

$$\cos\theta = \frac{1^2 + 4^2 - 4^2}{2 \times 1 \times 4} \quad \boxed{B}$$

$$= \frac{1}{8}$$

Alternative Solution:

If $2x - 1 = 3x + 1 \rightarrow x = -2$ which is outside the domain from (i) so $2x - 1 \neq 3x + 1$
If $2x - 1 = 5 - x \rightarrow x = 2$ which is outside the domain from (i) so $2x - 1 \neq 5 - x$

If $5 - x = 3x + 1 \rightarrow x = 1$ which is in the domain from (i) $\boxed{A}$

When $x = 1,\ 5 - x = 3x + 1 = 4$ and $2x - 1 = 1$

Let D be the midpoint of AC. Since ΔABC is isosceles with B as the apex, $BD \perp AC$.

$$\therefore \cos\theta = \frac{AD}{AB}$$

$$= \frac{\frac{1}{2} \times AC}{AB}$$

$$= \frac{0.5}{4} \quad \boxed{B}$$

$$= \frac{1}{8}$$

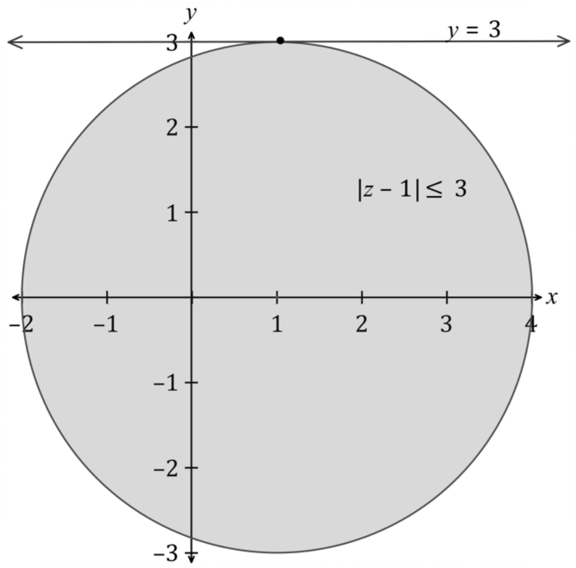
Question 16 (c)

Criteria	Marks
• Provides correct solution	2
• Correctly graphs both inequalities.	1

Sample answer:

$|z-1| \leq 3$ represents a region with centre (1, 0) and radius less than or equal to 3.
$\text{Im}(z) \geq 3$ represents a region above the horizontal line $y = 3$.

$\boxed{A}$



The point (1,3) is where the two inequalities hold. $\boxed{B}$

Question 16 (d) (i)

Criteria	Marks
• Provides correct solution	4
• Shows that if $m^2 + n^2 + p^2 = r^2 + R^2$ then the circle C is on S_3.	3
• Shows that if $m^2 + n^2 + p^2 = r^2 + R^2$ that ΔAPB is right angled	2
• Finds $d^2 = m^2 + n^2 + p^2$, or equivalent merit	1

Sample answer:

$d^2 = \left|\overrightarrow{AB}\right|^2 = (a + m - a)^2 + (b + n - b)^2 + (c - p - c)^2$

$= m^2 + n^2 + p^2 \qquad (1) \quad \boxed{A}$

Case 1: If $m^2 + n^2 + p^2 = r^2 + R^2$
From (1): $\qquad d^2 = r^2 + R^2 \qquad (2)$

Let P be any point on C.

From (2): $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AP}\right|^2 + \left|\overrightarrow{PB}\right|^2$ so ΔAPB is right angled at P $\quad \boxed{B}$

$\therefore \overrightarrow{AP} \cdot \overrightarrow{PB} = 0$
$\therefore$ P lies on the sphere S_3 with diameter AB

Since P is any point on C then C lies on S_3 if $m^2 + n^2 + p^2 = r^2 + R^2$. $\quad \boxed{C}$

Case 2: If $m^2 + n^2 + p^2 \neq r^2 + R^2$
From (1): $\qquad d^2 \neq r^2 + R^2 \qquad (3)$

$\therefore$ $\left|\overrightarrow{AB}\right|^2 \neq \left|\overrightarrow{AP}\right|^2 + \left|\overrightarrow{PB}\right|^2$ so ΔAPB is not right angled at P

$\therefore \overrightarrow{AP} \cdot \overrightarrow{PB} \neq 0$
$\therefore$ P does not lie on the sphere S_3.

Since P is any point on C then C does not lie on S_3 if $m^2 + n^2 + p^2 \neq r^2 + R^2$. $\quad \boxed{D}$

$\therefore$ the circle C lies on the sphere S_3 whose diameter is AB

only if $m^2 + n^2 + p^2 = r^2 + R^2$

Question 16 (d) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$x^2 + y^2 + z^2 = 25$ is a sphere with centre (0,0,0) and radius 5, so $r = 5$.

$(x-3)^2 + (y-4)^2 + (z-12)^2 = 144$ is a sphere with centre $(3, 4, 12)$ and radius 12, so $m = 3, n = 4, p = 12$ and $R = 12$.

$m^2 + n^2 + p^2 = 3^2 + 4^2 + 12^2 = 169$

$r^2 + R^2 = 5^2 + 12^2 = 169$

$\therefore\ m^2 + n^2 + p^2 = r^2 + R^2 \qquad \boxed{A}$

So from (i) we can see that the two spheres intersect in a circle that lies on a sphere with a diameter whose ends are at the centres of the spheres.